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Generalized Noether Theorems in Canonical Formalism for Field Theories and Their Applications

Zi-ping Li[1]

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A generalization of Noether's first theorem in phase space for an invariant system with a singular Lagrangian in field theories is derived and a generalization of Noether's second theorem in phase space for a noninvariant system in field theories is deduced. A counterexample is given to show that Dirac's conjecture fails. Some preliminary applications of the generalized Noether second theorem to the gauge field theories are discussed. It is pointed out that for certain systems with a noninvariant Lagrangian in canonical variables for field theories there is also a Dirac constraint. Along the trajectory of motion for a gauge-invariant system some supplementary relations of canonical variables and Lagrange multipliers connected with secondary first-class constraints are obtained.

1. INTRODUCTION

Noether's theorems refer to the invariance of systems. The usual considerations are based on an examination of the Lagrangian in configuration space and the corresponding transformation expressed in terms of Lagrange's variables. For a system with a regular Lagrangian in classical mechanics, the invariance under a finite continuous group in terms of Hamilton's variables was discussed by Djukic (1974). Many physically important systems are described in terms of singular Lagrangian. A system with a singular Lagrangian is subject to an inherent phase space constraint (Dirac, 1964). A generalization of Noether's theorem in canonical variables for a system of finite degrees of freedom with a singular Lagrangian was given by Li and Li (1991) and a system with a singular higher-order Lagrangian was also

[1]CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, and Department of Applied Physics, Beijing Polytechnic University, Beijing 100022, China.

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considered (Li, 1991). Here the symmetry properties of constrained Hamiltonian systems for field theories are further investigated. Generalizations of Noether's first and second theorems in field theories are given, which are useful for analyzing Dirac's constraint. Some applications are discussed.

The paper is organized as follows. In Section 2 the generalized Noether first theorem in phase space (GNFTPS) for a constrained Hamiltonian system is derived. In Section 3 some comments on Dirac's conjecture are given, and a counterexample is provided with the aid of the GNFTPS, to show that Dirac's conjecture is invalid. In Section 4 the generalized Noether second theorem and corresponding generalized Noether identities in phase space (GNIPS) for a noninvariant system are deduced. Preliminary applications of the GNIPS to the Yang–Mills field theories is discussed. It is pointed out that for certain systems with noninvariant Lagrangians in canonical variables for field theories there is also a Dirac constraint. Although Dirac's conjecture in general is invalid, we do not know of gauge field theories to which Dirac's conjecture leads to the wrong results. Sections 5 and 6 apply the GNIPS to the Abelian and non-Abelian gauge theories, respectively; along the trajectory of motion for gauge-invariant systems some supplementary relations of canonical variables and Lagrange multipliers connected with secondary first-class constraints are obtained. Finally, Section 7 is devoted to the conclusions.

2. GENERALIZED NOETHER FIRST THEOREM IN PHASE SPACE

Consider a system described by the state functions $\psi^\alpha(z)$ ($\alpha = 1, 2, \ldots, N$). In classical mechanics α is the index of generalized coordinates, and in field theories α is the index of the component of the field variables; $x=(t, \mathbf{r})$. The Lagrangian of the system depends on the set of state functions $\psi^\alpha(x)$ and their first-order derivatives: $\mathscr{L}(\psi^\alpha, \psi^\alpha_{,\mu})$, $\psi^\alpha_{,\mu} \equiv \partial_\mu \psi^\alpha$ ($\mu = 0, 1, 2, 3$). The flat space-time metric is $\eta_{\mu\nu} = \mathrm{diag}(+\,-\,-\,-)$. For many interesting physical systems the Lagrangian is singular, i.e., the Hessian matrix $H_{\alpha\beta}$ is degenerate,

$$\det(H_{\alpha\beta}) = \det\left(\frac{\partial^2 \mathscr{L}}{\partial \dot{\psi}^\alpha\, \partial \dot{\psi}^\beta}\right) = 0 \tag{1}$$

The Legendre transformation introduces canonical momenta $\pi_\alpha = \partial \mathscr{L}/\partial \dot{\psi}^\alpha$; one can then go over from the Lagrangian description to the Hamiltonian description. Suppose that the rank of the Hessian matrix is $N-R$; then one cannot solve for all $\dot{\psi}^\alpha$ from the definition of canonical momenta, because of (1). This implies the existence of constraints

$$\phi_a^0(\psi^\alpha, \pi_\alpha)=0 \qquad (a=1,2,\ldots,R) \tag{2}$$

The equations of motion of the constrained Hamiltonian system are given by

$$\dot{\psi}^\alpha=\delta H_T/\delta\pi_\alpha=\{\psi^\alpha, H_T\}, \qquad \dot{\pi}_\alpha=-\delta H_T/\delta\psi^\alpha=\{\pi_\alpha, H_T\} \tag{3}$$

where $H_T=\int_V d_x^3(\mathscr{H}_c+\lambda^a\phi_a^0)$, the λ^a are Lagrange multipliers, and $\{\cdot\,,\cdot\}$ denotes the Poisson brackets in field theories.

Suppose that for the system it is possible to construct a Lagrangian $\mathscr{L}_p$ in Hamilton's form and that the corresponding action integral is given by

$$I=\int_\Omega \mathscr{L}_p\, d^4x=\int_\Omega (\dot{\psi}^\alpha\pi_\alpha-\mathscr{H}_c)\, d^4x \tag{4}$$

where $\mathscr{H}_c(\psi^\alpha, \pi_\alpha)$ is the canonical Hamiltonian of the system. The summation is taken over repeated indices. Let us consider the transformation properties of the system under the infinitesimal transformation of the coordinates and canonical variables:

$$\begin{aligned} x_\mu \to x'_\mu &= x_\mu+\varepsilon_\sigma\tau^{\mu\sigma}(x, \psi^\alpha, \pi_\alpha)\\ \psi^\alpha(x)\to\psi^{\alpha'}(x') &= \psi^\alpha(x)+\varepsilon_\sigma\xi^{\alpha\sigma}(x, \psi^\alpha, \pi_\alpha)\\ \pi_\alpha(x)\to\pi'_\alpha(x') &= \pi_\alpha(x)+\varepsilon_\sigma\eta_\alpha^\sigma(x, \psi^\alpha, \pi_\alpha)\end{aligned} \tag{5}$$

generated by a finite continuous group of transformation G_r with constant parameters ε_σ $(\sigma=1,2,\ldots,r)$. Under this transformation suppose the change of Lagrangian $\mathscr{L}_p$ is invariant up to a divergence term, i.e., $\delta\mathscr{L}_p=\partial_\mu\Lambda^\mu=\varepsilon_\sigma\partial_\mu\Lambda^{\mu\sigma}(x, \psi^\alpha, \pi^\alpha)$; then one has

$$\frac{\delta I}{\delta\pi_\alpha}\,\bar{\delta}\pi_\alpha+\frac{\delta I}{\delta\psi^\alpha}\,\bar{\delta}\psi^\alpha+\partial_\mu[(\dot{\psi}^\alpha\pi_\alpha-\mathscr{H}_c)\,\delta x^\mu]+\frac{d}{dt}(\pi_\alpha\,\bar{\delta}\psi^\alpha)=\partial_\mu\Lambda^\mu \tag{6}$$

where

$$\frac{\delta I}{\delta\pi_\alpha}=\dot{\psi}^\alpha-\frac{\delta H_c}{\delta\pi_\alpha}, \qquad \frac{\delta I}{\delta\psi^\alpha}=-\dot{\pi}_\alpha-\frac{\delta H_c}{\delta\psi^\alpha} \tag{7}$$

and the local variations $\bar{\delta}\psi^\alpha$ and $\bar{\delta}\pi_\alpha$ can be expressed in terms of the total $\delta\psi^\alpha$ and $\delta\pi_\alpha$, respectively:

$$\bar{\delta}\psi^\alpha=\delta\psi^\alpha-\psi^\alpha_{,\mu}\,\delta x^\mu, \qquad \bar{\delta}\pi_\alpha=\delta\pi_\alpha-\pi_{\alpha,\mu}\,\delta x^\mu \tag{8}$$

Suppose that under the transformation (5) the change of the constraint conditions is given by $\delta\phi_a^0 = K_a = \varepsilon_\sigma K_a^\sigma$, hence

$$\frac{\partial\phi_a^0}{\partial\pi_\alpha}\bar{\delta}\pi_\alpha + \frac{\partial\phi_a^0}{\partial\psi^\alpha}\bar{\delta}\psi^\alpha = K_a - \frac{\partial\phi_a^0}{\partial\pi_\alpha}\pi_{\alpha,\mu}\,\delta x^\mu - \frac{\partial\phi_a^0}{\partial\psi^\alpha}\psi^\alpha_{,\mu}\,\delta x^\mu$$
$$\equiv F_a = \varepsilon_\sigma F_a^\sigma \tag{9}$$

Introducing the Lagrange multipliers $\lambda^a(x)$ and combining the expressions (6) and (9), one obtains by using the equations of motion (3)

$$\partial_\mu[(\dot{\psi}^\alpha\pi_\alpha - \mathscr{H}_c)\,\delta x^\mu - \Lambda^\mu] + \frac{d}{dt}(\pi_\alpha\,\bar{\delta}\psi^\alpha) = \lambda^a F_a \tag{10}$$

Consequently, the GNFTPS for a singular Lagrangian can be formulated in the following way: If under the transformation (5) the Lagrangian is invariant up to a divergence term and the constraint conditions are invariant under the local variations of the canonical variables, then

$$\int_V d^3x\,(\pi_\alpha\xi^{\alpha\sigma} - \mathscr{H}_c\tau^{0\sigma} - \Lambda^{0\sigma}) = \text{const} \qquad (\sigma = 1, 2, \ldots, r) \tag{11}$$

This result is a generalization of a regular and singular Lagrangian system with finite degrees of freedom for field theories (Djukic, 1974; Li and Li, 1991; Li, 1991).

3. DIRAC'S CONJECTURE

In the formulation of a dynamical system with a singular Lagrangian, Dirac (1964) conjectured that all first-class constraints (primary and secondary) are generators of gauge transformations. In turn, this problem is closely related to the equivalence of Dirac's procedure using the extended Hamiltonian H_E with the Lagrangian description. From time to time there have been objections to Dirac's conjecture (Sagano and Kamo, 1982; Castellani, 1982; Di Stefano, 1983; Costa *et al.*, 1985, Grácia and Pons, 1988). All of these objections are based on the observation that the equations of motion derived from an extended Hamiltonian are not strictly equivalent to the corresponding Lagrange equations. On the other hand, the Lagrange equations of motion can be recovered from the canonical equations derived from a total Hamiltonian H_T. This fact has led many authors (Castellani, 1982; Sugano, 1982; Sugano and Kimura, 1983*a–c*) to reject the extended Hamiltonian H_E as the generator of a reliable dynamical picture and to advocate the total Hamiltonian H_T as the correct time-development generator. Several examples have been given that Dirac's conjecture is not necessarily true (Allcock, 1975; Cawley, 1979, 1980; Frenkel, 1980). The results which were given by those examples were obtained from an improper linearization of the functional forms of the secondary first-class constraint. If one writes the constraint in linearized form, then this leads to a confusion of the concepts of weak and strong equality (Dirac, 1964), since the constraint $\psi \approx 0$ implies $\psi^2 \simeq 0$. Recently, it was pointed out by Qi (1990) that Dirac's conjecture holds in the examples given by Cawley and others.

Here this problem will be discussed from another point of view. Based on the symmetry properties of the system, let us consider whether the conservation laws derived from H_E via the GNFTPS (11) are equivalent to the results arising from the Lagrangian formalism via the classical Noether theorem. We presented an example (Li and Li, 1991) in which we did not write the constraint in linearized form and Dirac's conjecture was invalid. The conservation law was obtained formally via Noether's theorem in that example. It is easy to verify that this conservation law is trivially equal to zero according to the equations of motion. In addition, if one writes the constraints in linearized form in that example, then the set of all the constraints becomes second class. From the point of view of Dirac's conjecture, this is just as well, since neither generates a gauge transformation. While the linearization of the constraints was accepted by several authors, there have been some objections. We present another example here to avoid these ambiguities. For the sake of simplicity, we consider a dynamical system with finite degrees of freedom. A model Lagrangian is given by

$$L=\dot{x}(\dot{z}_1^2+\dot{z}_2^2-\dot{z}_3^2-\dot{z}_4^2)+y(z_1^2+z_2^2-z_3^2-z_4^2) \tag{12}$$

The canonical momenta are given by

$$\begin{aligned} &p_x=\dot{z}_1^2+\dot{z}_2^2-\dot{z}_3^2-\dot{z}_4, \qquad p_y=0, \qquad p_{z_1}=2\dot{x}\dot{z}_1 \\ &p_{z_2}=2\dot{x}\dot{z}_2, \qquad p_{z_3}=-2\dot{x}\dot{z}_3, \qquad p_{z_4}=-2\dot{x}\dot{z}_4 \end{aligned} \tag{13}$$

The generalized velocity $\dot{x}$ can be represented by

$$\dot{x}=\tfrac{1}{2}p_x^{-1/2}(p_{z_1}^2+p_{z_2}^2-p_{z_3}^2-p_{z_4}^2)^{1/2} \tag{14}$$

The canonical Hamiltonian is given by

$$H_c=p_x^{1/2}(p_{z_1}^2+p_{z_2}^2-p_{z_3}^2-p_{z_4}^2)^{1/2}-y(z_1^2+z_2^2-z_3^2-z_4^2) \tag{15}$$

The primary constraint of this system is given by

$$\phi=p_y\approx 0 \tag{16}$$

where the sign $\approx$ (weak equality) means equality on the constraint hypersurface (Dirac, 1964). The total Hamiltonian is given by

$$H_T = H_c + \lambda\phi \tag{17}$$

where λ is a Lagrange multiplier. The stationarity condition of the primary constraint yields the secondary constraint

$$\chi_1 = z_1^2 + z_2^2 - z_3^2 - z_4^2 \approx 0 \tag{18}$$

Because

$$\{\chi_1, H_T\} = \frac{1}{\dot{x}}(z_1 p_{z_1} + z_2 p_{z_2} + z_3 p_{z_3} + z_4 p_{z_4}) \tag{19}$$

the stationarity condition of the secondary constraint χ_1 yields the secondary constraint

$$\chi_2 = z_1 p_{z_1} + z_2 p_{z_2} + z_3 p_{z_3} + z_4 p_{z_4} \approx 0 \tag{20}$$

as long as $x \neq \text{const}$. Since

$$\{\chi_2, H_T\} = 2\dot{x}p_x \tag{21}$$

the stationarity condition of the secondary constraint χ_2 yields the secondary constraint

$$\chi_3 = p_x \approx 0 \tag{22}$$

as long as $x \neq \text{const}$. The set of the constraints given by (16), (18), (20), and (22) is clearly first class and no more constraints are generated. The extended Hamiltonian is given by

$$H_E = H_T + \mu_1\chi_1 + \mu_2\chi_2 + \mu_3\chi_3 \tag{23}$$

where μ_1, μ_2, and μ_3 are Lagrange multipliers.

The Lagrangian given by (12) is invariant under the transformation

$$\begin{aligned} z_1' &= z_1 + z_2\theta, & z_2' &= -z_1\theta + z_2 \\ z_3' &= z_3 + z_4\theta, & z_4' &= -z_3\theta + z_4 \end{aligned} \qquad (\theta = \text{const}, \quad |\theta| \ll |) \tag{24}$$

From the classical Noether theorem in the Lagrangian formalism one can obtain the corresponding conservation law. According to the canonical momenta introduced in (13), let us go over from the Lagrangian description to the Hamiltonian description. The Lagrangian L_p in Hamilton's form is invariant under (24) and the corresponding transformation of canonical momenta (13). The primary constraint condition (16) is also invariant under such a transformation. From the GNFTPS one obtains the same conservation law as from the classical Noether theorem. But if we take into account the secondary first-class constraints in this problem, we cannot obtain this conservation law from the extended Hamiltonian H_E. Dirac's conjecture fails and there is no linearization of the constraint in this example.

4. GENERALIZED NOETHER IDENTITIES IN PHASE SPACE

Consider the transformation properties of a system in phase space with respect to an infinite continuous group which leads to the GNIPS. In quantum theory, the GNIPS identities correspond to the Ward identities.

Let us consider an infinitesimal transformation in extended phase space

$$\begin{aligned} x^{\mu'} &= x^\mu + R^\mu_\sigma \varepsilon^\sigma(x) \\ \psi^{\alpha'}(x') &= \psi^\alpha(x) + S^\alpha_\sigma \varepsilon^\sigma(x) \\ \pi'_\alpha(x') &= \pi_\alpha(x) + T_{\alpha\sigma}\varepsilon^\sigma(x) \end{aligned} \tag{25}$$

generated by an infinite continuous group of transformations $G_{\infty r}$, with arbitrary functions $\varepsilon^\sigma(x)$ ($\sigma = 1, 2, \ldots, r$) and their derivatives up to some fixed order, where R^μ_σ, S^α_σ, and $T_{\alpha\sigma}$ are linear differential operators,

$$\begin{aligned} R^\mu_\sigma &= a^{\mu\nu(j)}_\sigma \partial_{\nu(j)} \\ S^\alpha_\sigma &= b^{\alpha\nu(k)}_\sigma \partial_{\nu(k)} \\ T_{\alpha\sigma} &= c^{\nu(l)}_{\alpha\sigma} \partial_{\nu(l)} \end{aligned} \tag{26}$$

with

$$A^{\mu(n)}_\sigma = A^{\overbrace{\mu\nu\ldots\sigma\rho}^{n}}_\sigma, \qquad \partial_{\mu(n)} = \overbrace{\partial_\mu\partial_\nu\ldots\partial_\sigma\partial_\rho}^{n} \tag{27}$$

$a, b,$ and c are functions of x, ψ^α, and π_α. Under the transformation (25) suppose that the change of the Lagrangian $\mathscr{L}_p$ is given by

$$\delta\mathscr{L}_p = \partial_\mu \Lambda^\mu + W \tag{28}$$

with $\Lambda^\mu = \Lambda^\mu_\sigma \varepsilon^\sigma(x)$ and $W = U_\sigma \varepsilon^\sigma(x)$, where U_σ and Λ^μ_σ are linear differential operators,

$$\begin{aligned} U_\sigma &= u^{\mu(m)}_\sigma \partial_{\mu(m)} \\ \Lambda^\mu_\sigma &= v^{\mu\nu(n)} \partial_{\nu(n)} \end{aligned} \tag{29}$$

where u and v are functions of x, ψ^α, and π_α. From (4) and (25), one has

$$\int_\Omega d^4x\left[\frac{\delta I}{\delta\pi_\alpha}(T_{\alpha\sigma}-\pi_{\alpha,\mu}R^\mu_\sigma)\varepsilon^\sigma+\frac{\delta I}{\delta\psi^\alpha}(S^\alpha_\sigma-\psi^\alpha_{,\mu}R^\mu_\sigma)\varepsilon^\sigma-U_\sigma\varepsilon^\sigma\right]$$
$$=\int_\Omega d^4x\left\{\partial_\mu[\Lambda^\mu_\sigma\varepsilon^\sigma+(\mathscr{H}_c-\dot\psi^\alpha\pi_\alpha)R^\mu_\sigma\varepsilon^\sigma]\right.$$
$$\left.-\frac{d}{dt}[\pi_\alpha(S^\alpha_\sigma-\psi^\alpha_{,\mu}R^\mu_\sigma)\varepsilon^\sigma]\right\} \tag{30}$$

Since the $\varepsilon^\sigma(x)$ are arbitrary functions, one can choose $\varepsilon^\sigma(x)$ and their derivatives up to the required order to vanish on the boundary of the domain; then we can make the terms of the right-hand side of the identity (30) vanish. We then repeat the integration by parts of the remaining terms of this identity, after which we apply the fundamental lemma of the calculus of variation to conclude that the generalized Noether second theorem in phase space can be formulated as: If the change of the Lagrangian $\mathscr{L}_p$ is given by (28) under an infinite continuous group $G_{\infty r}$ involving derivatives up to some given order inclusive, then there exist r identity relations between the functional derivatives $\delta I/\delta\psi^\alpha$ and $\delta I/\delta\pi_\alpha$ and their derivatives up to some order,

$$\tilde T_{\alpha\sigma}\left(\frac{\delta I}{\delta\pi_\alpha}\right)-\tilde R^\mu_\sigma\left(\pi_{\alpha,\mu}\frac{\delta I}{\delta\pi_\alpha}\right)+\tilde S^\alpha_\sigma\left(\frac{\delta I}{\delta\psi^\alpha}\right)-\tilde R^\mu_\sigma\left(\psi^\alpha_{,\mu}\frac{\delta I}{\delta\psi^\alpha}\right)=\tilde U_\sigma(1)$$
$$(\sigma=1,2,\ldots,r) \tag{31}$$

where $\tilde R^\mu_\sigma$, $\tilde S^\alpha_\sigma$, $\tilde T_{\alpha\sigma}$, and $\tilde U_\sigma$ are the operators adjoint to R^μ_σ, S^α_σ, $T_{\alpha\sigma}$ and U_σ, respectively, defined by

$$\int_\Omega fR^\mu_\sigma g\,d^4x=\int_\Omega g\tilde R^\mu_\sigma f\,d^4x+[\cdot]_B \tag{32}$$

where f, g are functions defined on the domain Ω and $[\cdot]_B$ is a boundary term, and similar expressions hold for S^α_σ, $\tilde S^\alpha_\sigma$; $T_{\alpha\sigma}$, $\tilde T_{\alpha\sigma}$, and U^σ, $\tilde U^\sigma$ in (31). $\tilde U_\sigma(1)$ indicates the operators adjoint to unity. These identities are generalizations of our previous work for field theories in the case where the Lagrangian $\mathscr{L}_p$ is variant under the infinite continuous group $G_{\infty r}$ (Li and Li, 1991; Li, 1991). These identities (31) are called the GNIPS.

We apply the GNIPS to the Yang–Mills theories; since the gauge transformation in general can be expressed as

$$\begin{aligned} \delta x^\mu &= 0 \\ \delta\psi^\alpha(x) &= b^\alpha_\sigma \varepsilon^\sigma(x) + b^{\alpha\mu}_\sigma \partial_\mu \varepsilon^\sigma(x) \\ \delta\pi_\alpha(x) &= c_{\alpha\sigma}\varepsilon^\sigma(x) \end{aligned} \tag{33}$$

then, in the case of $G_{\infty r}$ invariance, the GNIPS becomes

$$c_{\alpha\sigma}\frac{\delta I}{\delta\pi_\alpha} + b^\alpha_\sigma \frac{\delta I}{\delta\psi^\alpha} = \partial_\mu\left(b^{\alpha\mu}_\sigma \frac{\delta I}{\delta\psi^\alpha}\right) \tag{34}$$

Thus, we have the identity relations between the functional derivatives $\delta I/\delta\psi^\alpha$, $\delta I/\delta\pi_\alpha$, and their derivatives, and this leads to a relation in the number of linearly independent functional derivatives $\delta I/\delta\psi^\alpha$ and $\delta I/\delta\pi_\alpha$.

In the case of $G_{\infty r}$ invariance under the transformation (33), one has the basic identity

$$\frac{\delta I}{\delta\pi_\alpha} c_{\alpha\sigma}\varepsilon^\sigma + \frac{\delta I}{\delta\psi^\alpha}(b^\alpha_\sigma\varepsilon^\sigma + b^{\alpha\mu}_\sigma\partial_\mu\varepsilon^\sigma) + \frac{d}{dt}[\pi_\alpha(b^\alpha_\sigma\varepsilon^\sigma + b^{\alpha\mu}_\sigma\partial_\mu\varepsilon^\sigma)] = 0 \tag{35}$$

Multiplying (34) by ε^σ and substructing the result from (35), one obtains

$$\partial_\mu\left(b^{\alpha\mu}_\sigma \frac{\delta I}{\delta\psi^\alpha}\varepsilon^\sigma\right) + \frac{d}{dt}[\pi_\alpha(b^\alpha_\sigma\varepsilon^\sigma + b^{\alpha\mu}_\sigma\partial_\mu\varepsilon^\sigma)] = 0 \tag{36}$$

The GNIPS (34) and the identity (36) are independent of whether the ψ^α are a solution of the canonical equations of the constrained Hamiltonian system. Consider a constrained Hamiltonian system whose equations of motion are derived from the Hamiltonian $H = H_c + H' = H_c + \int d^3x\, \lambda^b \phi_b$, where ϕ_b are first-class constraints; then, along the trajectory of motion, the GNIPS (34) becomes supplementary conditions of the form

$$c_{\alpha\sigma}\frac{\delta H'}{\delta\pi_\alpha} + b^\alpha_\sigma \frac{\delta H'}{\delta\psi^\alpha} = \partial_\mu\left(b^{\alpha\mu}_\sigma \frac{\delta H'}{\delta\psi^\alpha}\right) \tag{37}$$

These conditions may have nontrivial meaning. They constitute the supplementary conditions on canonical variables of fields and the Lagrange multipliers.

As is well known, a gauge-invariant system in the Lagrangian formalism has a Dirac constraint. Using the GNIPS (31), we can further show that for certain noninvariant systems there is also a Dirac constraint. Suppose that under the transformation (33), the change of the Lagrangian $\mathscr{L}_p$ is given by $\delta\mathscr{L}_p = \partial_\mu\Lambda^\mu + U_\sigma\varepsilon^\sigma$, where $U_\sigma = u_\sigma + u^\mu_\sigma\partial_\mu + u^{\mu\nu}_\sigma\partial_\mu\partial_\nu$, with the u_σ and u^μ_σ as functions of x, ψ^α, and π_α and the $u^{\mu\nu}_\sigma$ are functions of x and ψ^α. For example, some massive Yang–Mills field theories belong to this category (Zhao and Yan, 1978; Li, 1982). In these circumstances, the GNIPS (31) becomes

$$c_{\alpha\sigma}\frac{\delta I}{\delta\pi_\alpha}+b_\sigma^\alpha\frac{\delta I}{\delta\psi^\alpha}+\partial_\mu\left[b_\sigma^{\alpha\mu}\left(\dot{\pi}_\alpha+\frac{\delta H}{\delta\psi^\alpha}\right)\right]=u_\sigma-\partial_\mu u_\sigma^\mu+\partial_\mu\partial_\nu u_\sigma^{\mu\nu} \tag{38}$$

Since

$$\dot{\pi}_\alpha=\frac{\partial^2\mathscr{L}}{\partial\dot{\psi}^\alpha\,\partial\psi^\beta}\,\dot{\psi}^\beta+\frac{\partial^2\mathscr{L}}{\partial\dot{\psi}^\alpha\,\partial\dot{\psi}^\beta}\,\ddot{\psi}^\beta \tag{39}$$

substituting the expressions (39) into the identities (38) leads to terms containing third time derivatives of ψ^β which must cancel each other irrespective of other terms,

$$b_\sigma^{\alpha\mu}\,\frac{\partial^2\mathscr{L}}{\partial\dot{\psi}^\alpha\,\partial\dot{\psi}^\beta}\,\dddot{\psi}^\beta=0 \tag{40}$$

These conditions are fulfilled for any third-order time derivative of ψ^α; therefore one obtains

$$b_\sigma^{\alpha\mu}H_{\alpha\beta}=0 \tag{41}$$

Since $b_\sigma^{\alpha\mu}$ are not all identically zero (for example, the gauge transformation), this implies $\det(H_{\alpha\beta})=0$; then the Hessian matrix is degenerate, and the system has a Dirac constraint.

In the following we give some applications of the GNIPS to gauge field theories.

5. ABELIAN CASE

First, let us consider the electromagnetic field coupled to a charged boson field whose Lagrangian is given by

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}[(\partial_\mu+ieA_\mu)\varphi]^*[(\partial_\mu+ieA_\mu)\varphi]-V(\varphi\varphi^*) \tag{42}$$

$$V(\varphi\varphi^*)=\mu\varphi\varphi^*+\tfrac{1}{2}\lambda(\varphi\varphi^*)^2 \tag{43}$$

For $\mu>0$ and $\lambda\neq 0$, the bosons are massive and self-interacting; for $\mu<0$ and $\lambda>0$, $V(\varphi\varphi^*)$ is called a Higgs potential. This Lagrangian (42) is invariant under the local $U(1)$ gauge transformation. If one writes $\varphi=\eta e^{i\theta}$ and use the modulus η and phase θ as new fields instead of the fields φ and φ^*, the Lagrangian expressed in terms of these new fields then becomes

$$\begin{aligned}\mathscr{L}=&-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta)-\tfrac{1}{2}\eta^2(\partial_\mu\theta)(\partial^\mu\theta)\\&-V(\eta)+A_\mu j^\mu+\tfrac{1}{2}e\eta^2A_\mu A^\mu\end{aligned}\tag{44}$$

with

$$j^\mu=-e\eta^2(\partial^\mu\theta+eA^\mu)$$

The Lagrangian (44) is invariant under the local transformation

$$\begin{aligned}A_\mu(x)&\to A'_\mu(x)=A_\mu(x)+\partial_\mu\varepsilon(x)\\\eta(x)&\to\eta'(x)=\eta(x)\\\theta(x)&\to\theta'(x)=\theta(x)+e\varepsilon(x)\end{aligned}\tag{45}$$

The canonical momenta are given by

$$\pi^\mu=\partial\mathscr{L}/\partial\dot{A}_\mu=-F^{0\mu}\tag{46a}$$

$$\pi_\eta=\partial\mathscr{L}/\partial\dot{\eta}=\partial_0\eta\tag{46b}$$

$$\pi_\theta=\partial\mathscr{L}/\partial\dot{\theta}=\eta^2(\partial_0\theta+eA_0)\tag{46c}$$

exhibiting the primary constraint

$$\Phi=\pi^0\approx0\tag{47}$$

The canonical Hamiltonian density is given by

$$\begin{aligned}\mathscr{H}=&\tfrac{1}{2}\pi_i\pi_i-A_0\partial_i\pi_i+\tfrac{1}{4}F_{ik}F_{ik}+\tfrac{1}{2}\pi_\eta^2+\frac{1}{2\eta^2}\pi_\theta^2-eA_0\pi_\theta\\&+\tfrac{1}{2}\eta^2(\partial_k\theta)(\partial_k\theta)+\tfrac{1}{2}(\partial_k\eta)(\partial_k\eta)+V(\eta)+e\eta^2A_k(\partial_k\theta+\tfrac{1}{2}eA_k)\end{aligned}\tag{48}$$

The requirement that $\{\pi^0, H_T\}$ vanishes at least weakly leads to the secondary constraint

$$\chi=\partial_i\pi_i+e\pi_\theta\approx0\tag{49}$$

where $H_T=\int d^3x\,(\mathscr{H}+\lambda\pi^0)$, with a multiplier function $\lambda(x)$. There are no further constraints. The set of constraints ($\Phi\approx0$ and $\chi\approx0$) is clearly first class. The Lagrangian $\mathscr{L}_p$ in canonical form is invariant under the local transformation

$$\begin{aligned}&\delta A_\mu=S_\mu\varepsilon(x)=\partial_\mu\varepsilon(x),\qquad\delta\eta=0,\qquad\delta\theta=S\theta(x)=e\varepsilon(x)\\&\delta\pi^\mu=0,\qquad\delta\pi_\eta=0,\qquad\delta\pi_\theta=T\varepsilon(x)=2e\eta^2\partial_0\varepsilon(x)\end{aligned}\tag{50}$$

The GNIPS (31) becomes

$$\tilde{T}\left(\dot{\theta}-\frac{\delta H}{\delta \pi_\theta}\right)+\tilde{S}_\mu\left(-\dot{\pi}^\mu-\frac{\delta H}{\delta A_\mu}\right)+\tilde{S}\left(-\dot{\pi}_\theta-\frac{\delta H}{\delta \theta}\right)=0 \qquad (51)$$

For a gauge-invariant system Dirac's conjecture is valid (Costa *et al.*, 1985). The dynamics of a system possessing constraints (47) and (49) should be described by the equations of motion arising from the extended Hamiltonian

$$H_E=H+H'=\int d^3x\,[\mathscr{H}+\lambda\pi^0+\mu(\partial_i\pi_i+e\pi_\theta)] \qquad (52)$$

where $\lambda(x)$ and $\mu(x)$ are multiplier functions. Along the trajectory of motion, the GNIPS (51) becomes

$$\tilde{T}\left(\frac{\delta H'}{\delta \pi_\theta}\right)+\tilde{S}_\mu\left(\frac{\delta H'}{\delta A_\mu}\right)+\tilde{S}\left(\frac{\delta H'}{\delta \theta}\right)=0 \qquad (53)$$

From the expressions (50), (52), and (53), one obtains

$$\partial_0[\mu(x)\eta^2(x)]=0 \qquad (54)$$

which implies that the multiplier $\mu(x)$ connected to the secondary first-class constraint must make the quantity $\mu(x)\eta^2(x)$ independent of time explicitly.

For the free electromagnetic field, Dirac's conjecture also holds; along the trajectory of the motion, the GNIPS becomes a trivial equality.

For the Proca field A_μ coupling with an external source j^μ, the Lagrangian is given by

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}m^2A_\mu A^\mu j^\mu \qquad (55)$$

One can proceed in the same way, under the transformation

$$\delta A^\mu=\partial^\mu\varepsilon(x), \qquad \delta\pi_\mu=0 \qquad (56)$$

where the π_μ are canonical momenta conjugate to A^μ; the GNIPS (31) implies the supplementary condition $m^2\partial_\mu A^\mu+\partial_\mu j^\mu=0$ along the trajectory of the motion. Then, the conservation of current j^μ is equivalent to the Lorentz condition. For an electromagnetic field coupling with an external source this implies that the electric charge density and the current density satisfy the equation of continuity.

6. NON-ABELIAN CASE

The Lagrangian of the Yang–Mills field coupled to a scalar field is given by

$$\mathscr{L} = -\tfrac{1}{4}[F^{\alpha}_{\mu\nu}]^2 - \tfrac{1}{2}[(D_\mu\varphi)^\alpha]^2 - V(\varphi) \tag{57}$$

where

$$F^{\alpha}_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + gc^\alpha_{\beta\gamma}A^\beta_\mu A^\gamma_\nu \tag{58}$$

and A^α_μ is a Yang–Mills field with non-Abelian group index α belonging to the adjoint representation of the non-Abelian gauge group G and $c^\alpha_{\beta\gamma}$ is the antisymmetric structure constant of the group. The scalar (Higgs) field φ^α belongs to an orthogonal representation of the group with

$$(D_\mu\varphi)^\alpha = \partial_\mu\varphi^\alpha + gA^\gamma_\mu I^\gamma_{\alpha\beta}\varphi^\beta \tag{59}$$

where the antisymmetric matrices I^γ represent the generator of the representation. The potential $V(\varphi)$ is a fourth-order polynomial in components φ^α, invariant under the action of G.

The canonical momenta are given by

$$\pi^\mu_\alpha = \partial\mathscr{L}/\partial\dot{A}^\alpha_\mu = -F^{0\mu}_\alpha \tag{60a}$$

$$\pi_\alpha = \partial\mathscr{L}/\partial\dot{\varphi}^\alpha = (D^0\varphi)_\alpha \tag{60b}$$

respectively. The Yang–Mills Lagrangian is also singular, and the primary constraints are

$$\Phi_\alpha = \pi^0_\alpha \approx 0 \tag{61}$$

The canonical Hamiltonian density is given by

$$\begin{aligned}\mathscr{H} = {} & \tfrac{1}{2}\pi^\alpha_i\pi^\alpha_i + \tfrac{1}{2}\pi^\alpha\pi^\alpha + \tfrac{1}{4}(F^\alpha_{ij})^2 + \tfrac{1}{2}[(D_i\varphi)^\alpha]^2 + V(\varphi) \\ & - A_0(\partial_i\pi^\alpha_i + gc^\gamma_{\alpha\beta}A^\beta_i\pi^i_\gamma - g\varphi^\beta I^\alpha_{\beta\gamma}\pi^\gamma)\end{aligned} \tag{62}$$

The consistency conditions $\{\pi^0_\alpha, H_T\} \approx 0$ lead to the secondary constraints

$$\chi^\alpha = \partial_i\pi_i + gc^\gamma_{\alpha\beta}A^\beta_i\pi^\gamma_i - g\varphi^\beta I^\alpha_{\beta\gamma}\pi^\gamma \approx 0 \tag{63}$$

where $H_T = \int d^3x\,(\mathscr{H} + \lambda^\alpha\pi^0_\alpha)$ with the multiplier functions $\lambda^\alpha(x)$. There is no further constraint. The set of constraints (61) and (63) is first class. The Lagrangian $\mathscr{L}_p$ is invariant under the transformation

$$\begin{aligned}
&\delta\varphi^\alpha = S^\alpha_\beta\varepsilon^\beta(x) = -igI^\beta_{\alpha\sigma}\varphi^\sigma\varepsilon^\beta(x) \\
&\delta\pi^\sigma = T^\alpha_\beta\varepsilon^\beta(x) = (\partial_0 S^\alpha_\beta + gA^\gamma_0 I^\gamma_{\alpha\sigma}S^\alpha_\beta\partial_0)\varepsilon^\beta(x) \\
&\delta A^\alpha_\mu = D^\alpha_{\beta\mu}\varepsilon^\beta(x) \qquad (D^\alpha_{\beta\mu} = \delta^\alpha_\beta\partial_\mu + gc^\alpha_{\beta\gamma}A^\gamma_\mu) \\
&\delta\pi^\mu_\alpha = T^\mu_{\alpha\beta}\varepsilon^\beta(x) \qquad (T^\mu_{\alpha\beta} = gc^\alpha_{\beta\gamma}\pi^\mu_\eta)
\end{aligned} \tag{64}$$

For a gauge-invariant system, the dynamics of the system can be described by the equations of motion arising from the extended Hamiltonian; along the trajectory of motion the GNIPS (31) becomes

$$\tilde{T}^{\alpha}_{\beta}\left(\frac{\delta H'}{\delta \pi^{\alpha}}\right)+\tilde{S}^{\alpha}_{\beta}\left(\frac{\delta H'}{\delta \varphi^{\alpha}}\right)+\tilde{T}^{\mu}_{\alpha\beta}\left(\frac{\delta H'}{\delta \pi^{\mu}_{\alpha}}\right)+\tilde{D}^{\alpha}_{\beta\mu}\left(\frac{\delta H'}{\delta A^{\alpha}_{\mu}}\right)=0 \tag{65}$$

where

$$H'=\int d^3x\,[\lambda^{\alpha}\pi^0_{\alpha}+\mu_{\alpha}(\partial_i\pi^{\alpha}_i+gc^{\gamma}_{\alpha\beta}A^{\beta}_i\pi^i_{\gamma}-g\varphi^{\beta}I^{\alpha}_{\beta\gamma}\pi^{\gamma}] \tag{66}$$

with the Lagrangian multiplier functions $\lambda^{\alpha}(x)$ and $\mu_{\alpha}(x)$. Substituting the expressions (64) and (66) into the expressions (65), one obtains

$$\begin{aligned}
&ig^2I^{\alpha}_{\gamma\sigma}I^{\beta}_{\delta\sigma}I^{\rho}_{\lambda\alpha}\mu_{\rho}A^{\gamma}_0\varphi^{\delta}\varphi^{\lambda}-igI^{\beta}_{\alpha\delta}I^{\rho}_{\alpha\delta}\varphi^{\delta}\partial_0(\mu_{\rho}\varphi^{\lambda})\\
&\quad+igI^{\beta}_{\alpha\sigma}I^{\rho}_{\alpha\gamma}\mu_{\rho}\varphi^{\delta}\pi^{\gamma}\\
&\quad+c^{\alpha}_{\beta\gamma}[\mu_{\alpha}\partial_i\pi^{\gamma}_i+gc^{\rho}_{\sigma\alpha}\mu_{\rho}(\pi^{\sigma}_iA^{\gamma}_i-\pi^{\gamma}_iA^{\sigma}_i)]=0
\end{aligned} \tag{67}$$

Thus, if Dirac's conjecture is accepted, then some supplementary relations of the canonical variables and Lagrangian multipliers connected with secondary first-class constraints are obtained for the model (57).

For the pure Yang–Mills field, Dirac's conjecture holds (Castellani, 1982); along the trajectory of the motion the GNIPS is

$$c^{\alpha}_{\beta\gamma}[\mu_{\alpha}\partial_i\pi^{\gamma}_i+gc^{\rho}_{\sigma\alpha}\mu_{\rho}(\pi^{\sigma}_iA^{\gamma}_i-\pi^{\gamma}_iA^{\sigma}_i)]=0$$

These supplementary conditions do not involve the derivative of the Lagrangian multipliers $\mu_{\alpha}(x)$ connected with secondary first-class constraints.

7. CONCLUSIONS

Based on the canonical formalism, the GNFTPS for an invariant system with a singular Lagrangian and the GNIPS for a noninvariant system in field theories were derived. Considering the symmetry properties of a system, a counterexample was shown on the invalidity of Dirac's conjecture for a constrained Hamiltonian system from the viewpoint of the generalized Noether theorem in the context of phase space. A preliminary application of the GNIPS to field theories was discussed. The GNIPS gives us some identity relations between the functional derivatives (with respect to canonical variables) and their derivatives. By combining the GNIPS and the constraint conditions, one can find more relationships among some of the variables. Using the GNIPS, it has been shown that for certain noninvariant systems in the canonical formalism there is also a Dirac constraint. Along the trajectory of the motion the GNIPS becomes the supplementary conditions on canonical variables and the Lagrange multipliers. The application of the GNIPS to models in Abelian and non-Abelian gauge field theories was discussed in detail. The application of the GNFTPS and the GNIPS to field theories enables us to obtain additional information about the Dirac constraint and the Lagrange multipliers.

The extension of the theory to field theories with singular higher-order Lagrangians is formally straightforward, but the extension of the theory to supersymmetry needs further consideration.

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